Wreath Product Decompositions

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Algebra and Representation Theory

RWTH AACHEN UNIVERSITY

Computational Group Theory

Groups encode symmetry

What is a group?

Groups encode symmetry

symmetries of square

rotations of tetrahedron

shuffles of cards

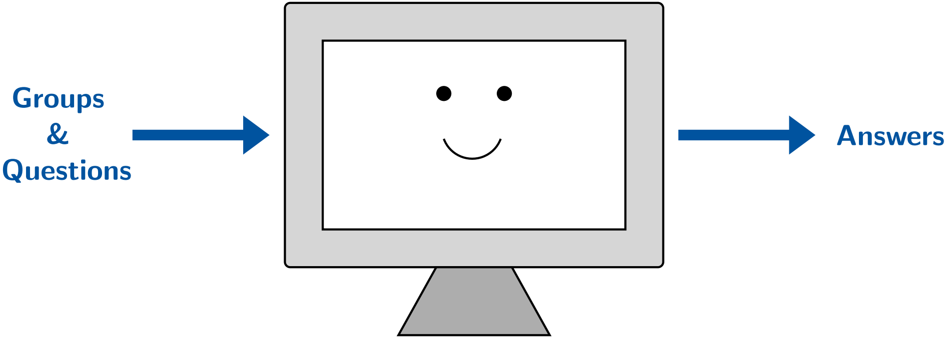
Groups encode symmetry

$\mathrm{Dih}(2 \cdot 4)$

$\mathrm{Alt}(4)$

$\mathrm{Sym}(4)$

The computer is our friend!



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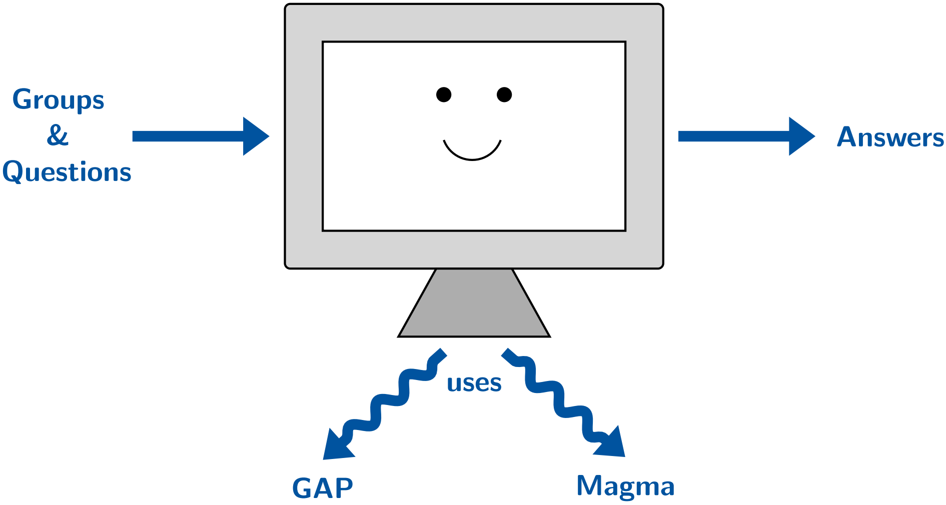

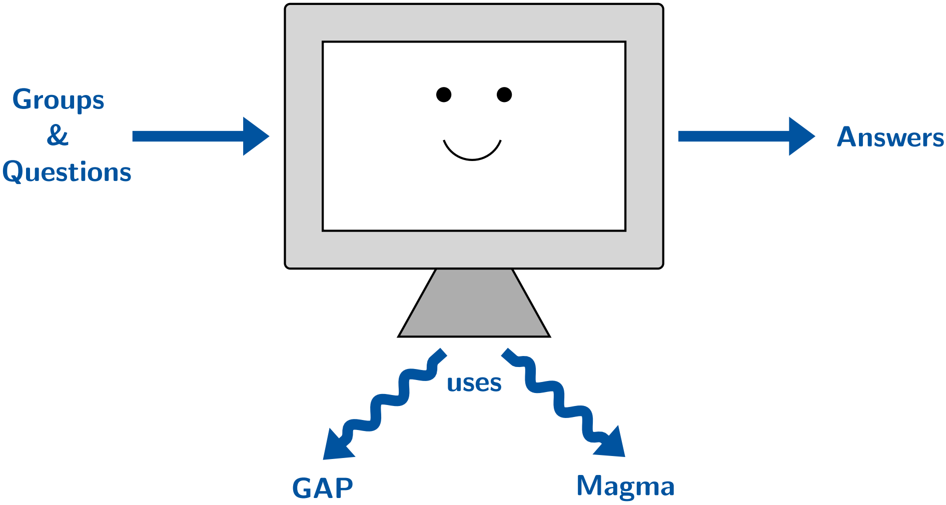

How does it work?

There are numerous ways to represent a (finite) group.

Our setting

$$\underbrace{P = \mathrm{Sym}(n)}_{\text{permutation group}} \quad \text{or} \quad \underbrace{P = \mathrm{GL}(d, q)}_{\text{matrix group}}$$

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Represent a group G via a **generating set** $X \subseteq P$ as

$$G = \langle X \rangle := \{\underbrace{x_1 \cdots x_\ell}_{\text{word}} : x_i \in X \cup X^{-1}, \ell \in \mathbb{N}\}.$$

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For example, $\mathrm{Sym}(n) = \langle\, (1,2), (1,2,\ldots,n) \,\rangle \rightsquigarrow \mathcal{O}(n \log(n))$ bits.
However, $|\mathrm{Sym}(n)| = n!$ is comparably very large.

There are various questions/problems posed on groups.

Fundamental problems

- **Group order:** Determine $|G|$.
- **Membership:** Given $x \in P$, is x an element of G?
- **Rewriting:** Given $g \in G$, write g as a word in X.

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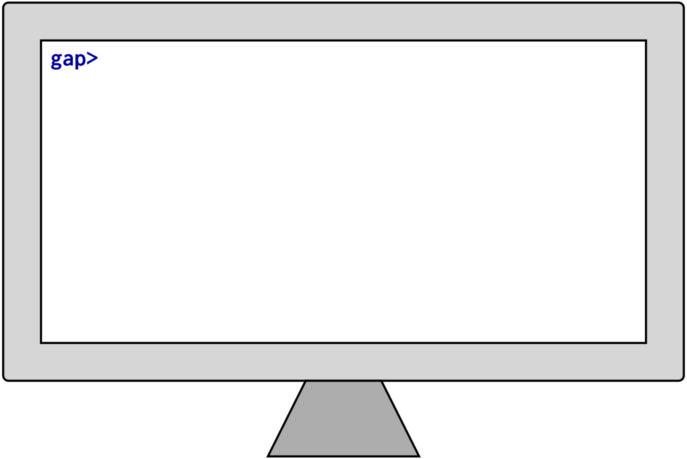
Solving these problems efficiently is a non-trivial task!

Talking to the computer



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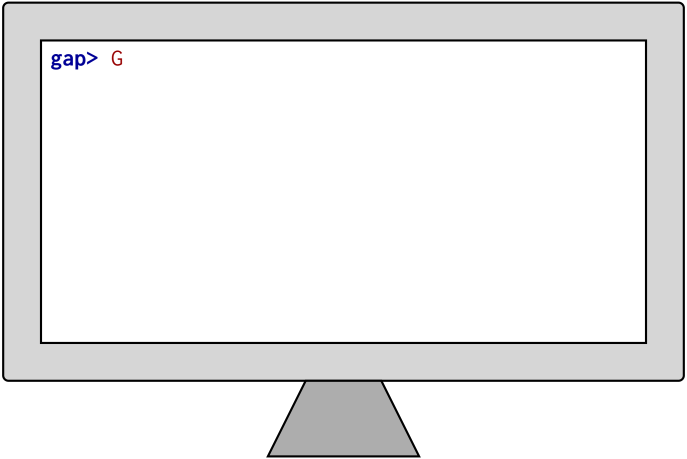

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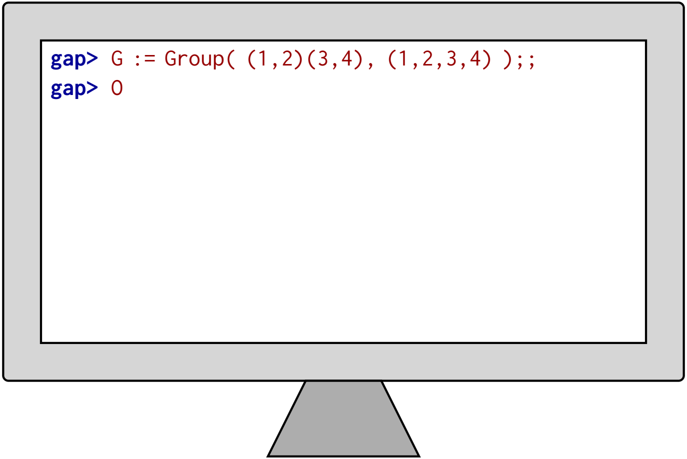

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gap> Order(G);
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```

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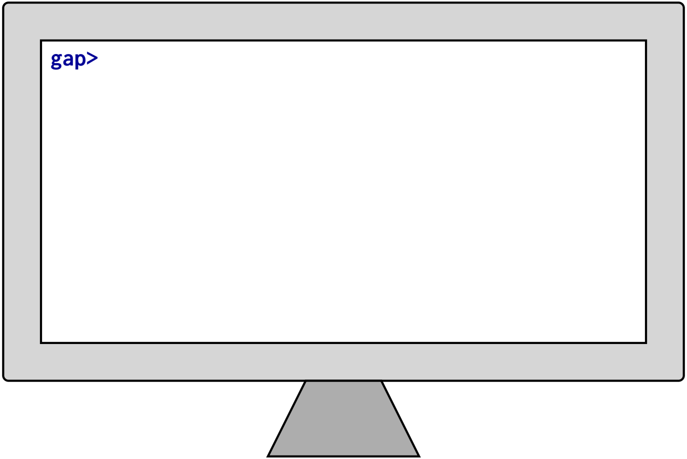

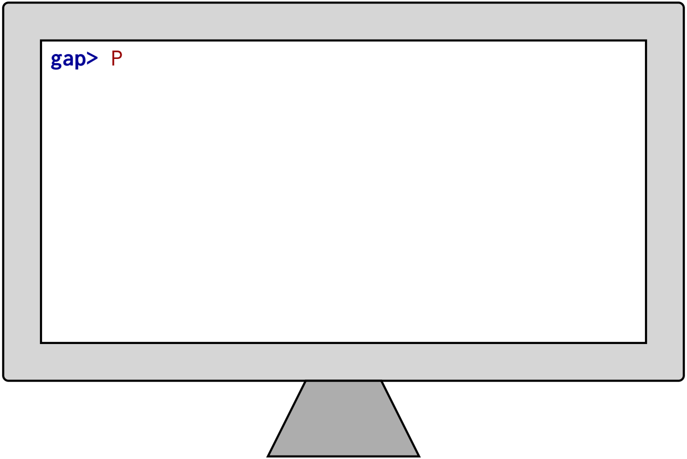
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```
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Talking to the computer

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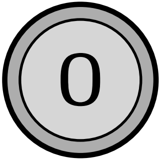
Randomisation is key

Random Elements



Random Elements

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Bits:

Input: $G = \langle X \rangle$ and $\ell \in \mathbb{N}$

Output: ℓ independent (nearly-)uniformly distributed elements from G

Randomisation is key

Random Elements

Bits: 1 0 0 1 0 1 1 1 ...

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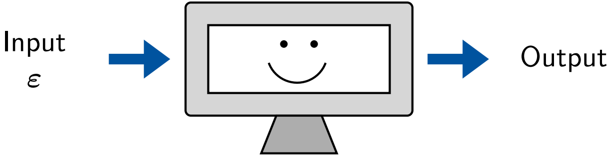
Monte-Carlo algorithms

Let $\varepsilon \in (0,1)$ be an error probability



$\mathrm{Prob}(\text{Output is incorrect}) \leq \varepsilon$

Runtime is $\mathcal{O}\big(f(\text{Input}) \cdot \log(1/\varepsilon)\big)$

Black box model

A computational model that covers all finite group representations

$$G = \langle X \rangle$$

$$\underbrace{\begin{matrix} 001000010 \\ 110001101 \\ 000100010 \\ \vdots \end{matrix}}_{\text{elements of } G}$$

A computational model that covers all finite group representations

allowed

x^{-1}

$x * y$

$x = y$

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not allowed

action on point

entry of a matrix

anything else...

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Input: $G = \langle X \rangle \leq \mathrm{Sym}(n)$

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$$G = G^{[1]} \geqslant G^{[2]} \geqslant \cdots \geqslant G^{[\beta]} \geqslant G^{[\beta+1]} = \{1_G\}\,,$$

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If G is **small-base**, then many algorithms are nearly-linear:

$$\mathcal{O}\left(n \log(n)^c \right)$$

If G is **large-base**, then many algorithms are polynomial:

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\begin{tikzcd}
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Leaf Nodes: (quasi-)simple groups

Computing in large matrix groups

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Leaf Nodes: (quasi-)simple groups $\rightsquigarrow$ **constructive recognition**

Theorem (Classification of finite simple groups (CFSG))

Let G be a finite simple group. Then G is one of the following:

- a cyclic group of prime order;
- a finite alternating group of degree at least 5;
- a finite group of Lie type;
- one of 26 sporadic groups.

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Output: $\varphi : G \to \overset{♛}{G}$ isomorphism to golden copy of G

	Matrix Groups	Permutation Groups
Input:	$G \leq \mathrm{GL}(d, q)$	$G \leq \mathrm{Sym}(n)$

How about large permutation groups?

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Wreath Products

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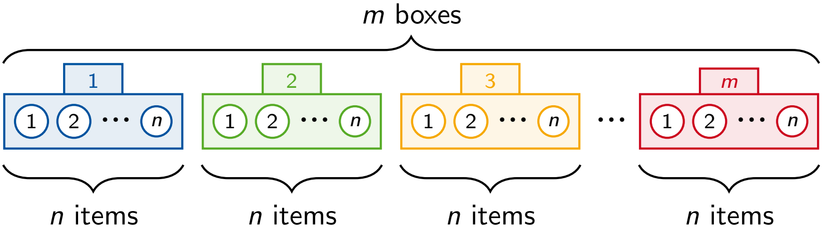

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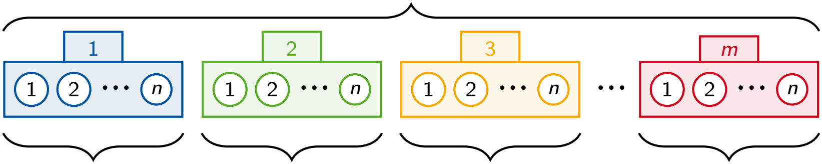

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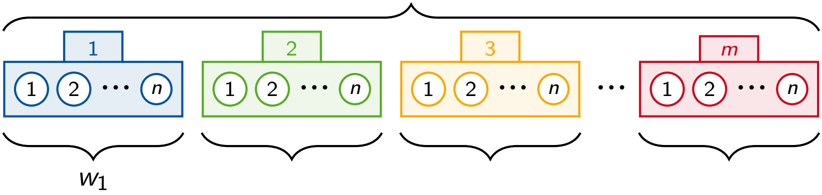

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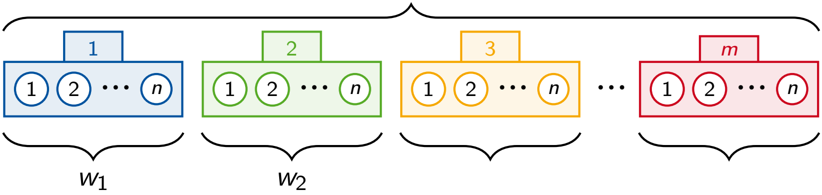

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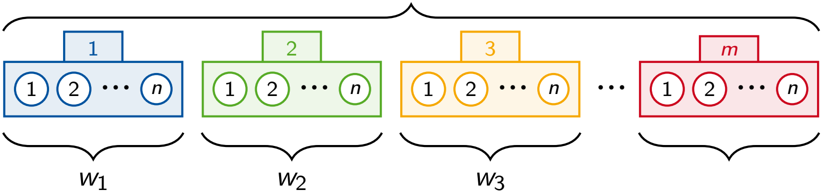

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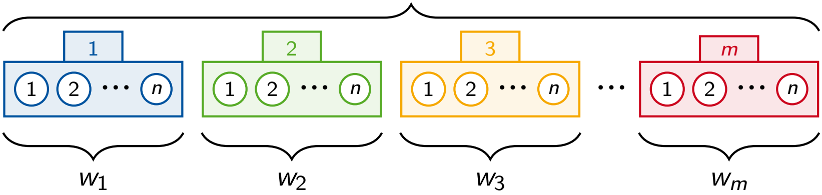

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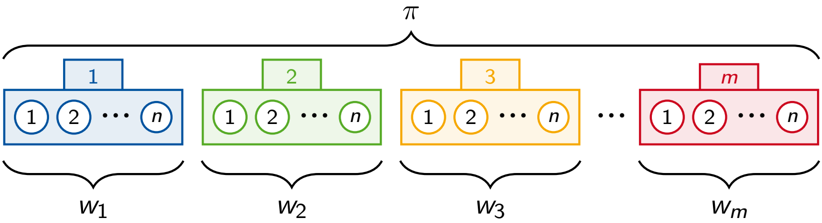

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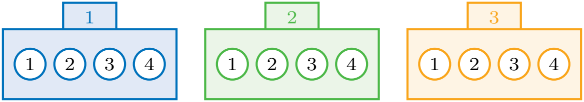

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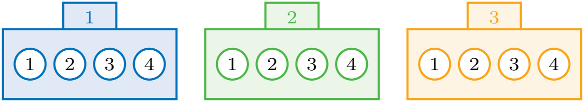

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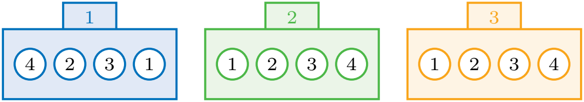

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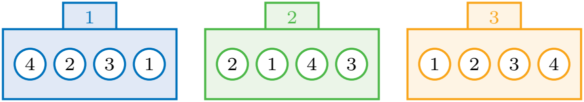

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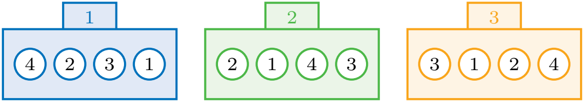

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Theorem (Liebeck, 84)

If $G \leq \mathrm{Sym}(N)$ be large-base primitive, then

$$\exists\, \pi \in \mathrm{Sym}(N) : \mathrm{Alt}(\ell)^m \leq G^\pi \leq \mathrm{Sym}(\ell) \wr \mathrm{Sym}(m) =: W,$$

where W acts in (PA) on $N = n^m$ points and $n = \binom{\ell}{k}$.

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Theorem (Liebeck, 84)

If $G \leq \mathrm{Sym}(N)$ be large-base primitive, then

$$\exists\, \pi \in \mathrm{Sym}(N) : \mathrm{Alt}(\ell)^m \leq G^\pi \leq \mathrm{Sym}(\ell) \wr \mathrm{Sym}(m) =: W,$$

where W acts in (PA) on $N = n^m$ points and $n = \binom{\ell}{k}$.

$\rightsquigarrow$ **Idea:** find π and translate from product action on (n^m) points into imprimitive action on $(n \cdot m)$ points.

Input: $G = \langle X \rangle$, a black box group such that
$\mathrm{Soc}(K)^m = \mathrm{Soc}(W) \lesssim G \lesssim W := K \wr \mathrm{Sym}(m)$,
K is **almost simple**, and
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Wreath product decomposition

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Theorem (R., 2025)

If $K = \mathrm{Sym}(n)$ or $K = \mathrm{Alt}(n)$ or $K = \mathrm{PSX}(d, q)$ with q odd, then there exists a polynomial-time* Monte-Carlo algorithm to compute a **wreath product decomposition**.

* under the assumption of a random element oracle and element order oracle

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- compression: (n^m) points $\to$ $(n \cdot m)$ points
- seeing components is computationally very useful!

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Strategy

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Input: $G = \langle X \rangle$

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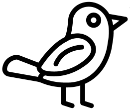
$\exists\ m \in \mathbb{N}$:

$T^m \lesssim G \lesssim T \wr \mathrm{Sym}(m)$

& G acts transitively

on base components



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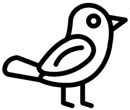

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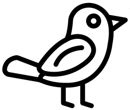

But what is T or m?

$$W := T \wr \mathrm{Sym}(m) = T_1 \times \cdots \times T_m \rtimes \mathrm{Sym}(m)$$

Steps of algorithm

Let $\varphi_0 : G \to W = T \wr \mathrm{Sym}(m)$ be an unknown embedding.

Step 1: Single-Component Group

Compute $S \leq G$ with $S^{\varphi_0} = T_1 \cong T$.

Step 2: Constructive Recognition

Compute isomorphism $\lambda : S \to T$

Step 3: Top Group Action

Compute $t_1, \ldots, t_m \in G$ with $(S^{t_i})^{\varphi_0} = T_i$.

Step 4: Embedding

Compute embedding $\varphi : G \to W$.

After a long reduction, the success probability depends on

$$P_2(K) := \frac{\left|\left\{(x,y) \in K^2 : |x|_2 \neq |y|_2\right\}\right|}{|K^2|}$$

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Theorem (R., 2025)

$$\begin{aligned} \text{Let } \mathscr{K} = \quad & \{\mathrm{Alt}(n) : n \geq 5\} \\ \cup\, & \{\mathrm{Sym}(n) : n \geq 5\} \\ \cup\, & \{\mathrm{SX}(d,q) : d > 1, q \text{ odd}\} \\ \cup\, & \{\mathrm{PSX}(d,q) : d > 1, q \text{ odd}\}. \end{aligned}$$

There exists an explicit constant $c > 0$ such that

$$\forall\, K \in \mathscr{K} : P_2(K) \geq c$$

Applications

Input: $G = \langle X \rangle$, a black box group such that
$\mathrm{Soc}(K)^m \cong \mathrm{Soc}(G) \leq G \cong K \wr H$,
K is almost simple, and
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Constructive recognition

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Output: $\varphi : G \to G$ isomorphism to natural wreath product
and we can solve fundamental problems for G.

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and we can solve fundamental problems for $\overset{👑}{G}$.

Theorem (R., 2025)

If $K = \mathrm{Sym}(n)$ or $K = \mathrm{Alt}(n)$ or $K = \mathrm{PSX}(d, q)$ with q odd, and if $H \le \mathrm{Sym}(m)$ transitive, then constructive recognition of $G = K \wr H$ can be reduced in polynomial-time* to constructive recognition of K and H.

* under the assumption of a random element oracle and element order oracle

Conjugacy-related problems

Results from [Bernhardt, Niemeyer, R., Wollenhaupt, '22]

Let $W := K \wr H$ and $H \leq \mathrm{Sym}(m)$. We describe algorithms ...

- to solve the **conjugacy problem** for two elements in W;
- to compute the **centraliser** of an element in W;
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Main Idea

Break down problems ...

- from wreath product elements onto **wreath cycles**;
- from W onto K and H.

Results from [Bernhardt, Niemeyer, ...

Shameless Advertisement

My algorithms in GAP:

For example, we can enumerate representatives of all **514 976** conjugacy classes in $\mathrm{Sym}(8) \wr \mathrm{Alt}(6)$ in around **5 seconds**,

or compute conjugating elements in $\mathrm{Sym}(25) \wr \mathrm{Alt}(100)$ in a few milliseconds

Thanks for your attention!